A Practical, Nondiverging Filter

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A Kalman filter, and in fact all filters based on the usual approaches, have the troublesome problem that although they are convergent if the models and the statistics of the disturbances are correctly known, divergence can result when such knowledge is not available. This paper describes a method to remedy this problem by weighing past observations by a progressively small number. The result is a filter with an algorithm which differs from the Kalman filter in just one additional multiplication by a fixed scalar at each observation time. Examples document the improvements obtainable with the modified filter.

1. Introduction

THE common underlying assumptions of basic filtering theory[1-10] are that the disturbance statistics, the system dynamics, and their variations in time are completely known and precisely modeled in the filter. Significantly, this assumption implies that observations taken at the present contribute only as pertinent information as those taken any time in the past. Based on these assumptions the Kalman filter is known to give, in principle, an increasingly accurate estimate as additional measurement data are processed. The norm of the covariance matrix of estimtion error is a monotonically decreasing function of the number of measurements. Hence new measurements have progressively less correcting effect on the estimates as they should in principle.

Estimation errors observed[11,12] in practical situations however tend to be much higher than predicted by the theory. Errors can even increase monotonically as additional measurements are being processed. This is known as the problem of the divergence of the filter estimates. It seems that the root cause of the difficulties with filtering is the unavoidable discrepancy between reality and its model and the accompanying degradaton of the information value of old observations. The basic filter equations do not account for such degradation since none exists as long as the basic filter assumptions are strictly satisfied. In reality unavoidable approximations leave many of the slower and smaller components of the dynamics unaccounted for. These collectively and in combination with other inaccuracies of modeling and computation cause a shifting uncertainty in the model and result in observations being significant only locally in time and becoming obsolete as time goes on.

The remedy proposed in this paper aims directly at accounting for this obsolescence of the observations. Its basis is weighted regression analysis and it is equivalent to the assumption that the covariance matrix of each past observation is escalating exponentially as time goes on. Statistical arguments for this could be constructed but it will be presented here simply as an arbitrary weighting to account for the obsolescence of past observations which is a practical fact.

Aside from the fact that this seems to aim at the root cause of the difficulties, the principal argument for this solution is its effectiveness, as illustrated by examples, and even more importantly its computational attractiveness. The algorithm to be presented differs from the basic Kalman filter algorithm in one single additional multiplication by a fixed scalar at each observation time. This scalar factor is empirically selected.

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2. Literature Survey

Heffes[13] and Nishimura[14] studied the effects on the Kalman filter performance of error in the knowledge of the covariance matrices describing the initial conditions of the system state vectors and the white measurement and plant noise vectors. Friedland[15] studied the effect on the covariance matrix of the Kalman filter due to applying incorrect gain in the filter. Neal,[16] and Huddle and Wismer[17] considered the performance degradation incurred by basing the filter design on incorrect models in the plant dynamics. Significantly, these studies could be used in determining the sensitivity of the estimator to model approximations.

For preventing filter divergence several approaches have beeen proposed. Schmidt,[18] Pines,[19] and Schlee,[11] have suggested modifications to the filter equations by modeling additional state variables (biases) and including their uncertainties in the filter without increasing the number of states to be estimated by the filter. In addition Pines[19] and Schlee[11] have developed a machine noise modification based on an assumed model of the errors caused by round-off in the digital computer. Holick et al[20] were investigating the technique of filter reset to keep the diagonal elements of the state covariance matrix above a specified value. Peschon and Larson[21] attempted to account for any unrealistic assumptions made about the system model by including a random variable at the model imput. The major reason for this approach is to allow the elements of the covariance matrix of the plant noise to be adjusted so that the steady-state filter gain matrix is nonzero. Schmidt[22] suggested two methods. One computes an estimate which is a linear combination of the estimate given all prior data with the estimate given no prior data. Past information is degraded. The other method assumes a priori lower bounds on certain projections of the covariance matrix. Jazwinski[12] proposed a filter which combines batch processing with recursive filtering. This filter requires memory varying between N and $2N$, where N is fixed a priori. Schmidt et al[34] designed the modified Kalman filter equations by including an additive gain matrix to the conventional gain matrix for the Kalman filter for preventing divergence of the filter resulting from unmodeled and computational errors.

All these methods are either based on some arbitrary modification thus nonrigorous or requiring some special storage thus nonfilter type operation but are generally justified by exhibiting satisfactory operation of the estimator.

A related problem to the dynamical error is the identification problem. Three major methods have been developed: 1) the instrumental variable methods[23,24]; 2) the linear least-square methods[25–27]; 3) the stochastic approximation and related techniques.[28–30] All these methods estimate only the parameters of the difference equation and not the covariance of the associated noises. All of them require knowledge of both input and output measurements. Kopp and Orford[31] applied quasi-linearization method to estimate the plant parameters. Judging from the simulation results,[31] this method is only effective for estimating linearly slowly varying parameters. Kashyap[32] applied maximum likelihood method to identify the system parameters. This method can only estimate the stationary linear systems. Kroy[33] obtained the analytical solution for the identification of a system parameter. The results are very complicated even for a scalar system.

3. Statement of the Problem

Given: A discrete time plant or signal model

$$\mathbf{x}_{n+1} = \mathbf{\Phi}(t_{n+1}, t_n)\mathbf{x}_n + \mathbf{u}_n,\ \mathbf{x}_n = \mathbf{x}(t_n),\ t_{n+1} - t_n = 1 \quad (1)$$

and the observation

$$\mathbf{y}_n = \mathbf{H}_n\mathbf{x}_n + \mathbf{w}_n \quad (2)$$

where $\mathbf{x}$ is the state vector $(i \times 1)$, $\mathbf{y}$ the observation vector $(j \times 1)$, $\mathbf{\Phi}(t_{n+1}, t_n)$ is the state transition matrix $(i \times i)$ and $\mathbf{H}_n$ is the observation matrix $(j \times i)$. The $\mathbf{\Phi}$, $\mathbf{H}$ system is observable. $\mathbf{u}_n$ and $\mathbf{w}_n$ are independent zero mean gaussian sequences

$$E\mathbf{u}_n = E\mathbf{w}_n = 0 \quad (3)$$

$$E\{\mathbf{u}_k\mathbf{u}_k' | t = t_n\} = \mathbf{Q}_k, \quad n \geq k \quad (4)$$

where E is the expectation operator

$$E\{\mathbf{w}_k\mathbf{w}_k' | t = t_n\} = \mathbf{R}_k|_n = s^{n-k}\mathbf{R}_k, \quad n \geq k \quad (5)$$

$$E\mathbf{u}_n\mathbf{u}_m' = E\mathbf{w}_n\mathbf{w}_m' = 0 \text{ if } m \neq n \text{ and } E\mathbf{u}_n\mathbf{w}_m' = 0 \quad (6)$$

where $s \geq 1$ is a scalar weighting factor.

Remark: The covariance of $\mathbf{w}_k$ defined in Eq. (5) presents the point of departure since it represents the covariance of a particular sample $\mathbf{y}_k$ taken at time t_k as increasing with time at a rate of s^{n-k} where n is the present time. This assumption represents the obsolescence of old measurements as explained elsewhere.

Find:

$$\hat{\mathbf{x}}_n = E\{\mathbf{x}_n | \mathbf{y}_k,\ 1 \leq k \leq n;\ \ \hat{\mathbf{x}}_1, \mathbf{P}_1|_1\} \quad (7)$$

that is find the expected value of the state $\mathbf{x}_n$ conditioned on the set of all past observations, $\mathbf{y}_k$, and on the knowledge of

$$E\mathbf{x}_1 = \hat{\mathbf{x}}_1, \quad E(\mathbf{x}_1 - \hat{\mathbf{x}}_1)(\mathbf{x}_1 - \hat{\mathbf{x}}_1)' = \mathbf{P}_1|_1 \quad (8)$$

$\mathbf{x}_1$ is assumed to be gaussian and independent of $\mathbf{u}_n$ and $\mathbf{w}_n$.

4. Solution of the Problem

$$\hat{\mathbf{x}}_{n+1} = \mathbf{\Phi}(t_{n+1}, t_n)\hat{\mathbf{x}}_n + \mathbf{P}_{n+1}|_n\mathbf{H}_n'(R_{n+1} + \mathbf{H}_n\mathbf{P}_{n+1}|_n\mathbf{H}_n')^{-1}[\mathbf{y}_{n+1} - \mathbf{H}_n\mathbf{\Phi}(t_{n+1}, t_n)\hat{\mathbf{x}}_n] \quad (9)$$

$$\mathbf{P}_{n+1}|_{n+1} = \mathbf{P}_{n+1}|_n - \mathbf{P}_{n+1}|_n\mathbf{H}_n'(\mathbf{R}_{n+1} + \mathbf{H}_n\mathbf{P}_{n+1}|_n\mathbf{H}_n')^{-1}\mathbf{H}_n\mathbf{P}_{n+1}|_n \quad (10)$$

$$\mathbf{P}_{n+1}|_n = s\mathbf{\Phi}(t_{n+1}, t_n)\mathbf{P}_n|_n\mathbf{\Phi}'(t_{n+1}, t_n) + \mathbf{Q}_n \quad (11)$$

$$\mathbf{P}_n|_n = E(\mathbf{x}_n - \hat{\mathbf{x}}_n)(\mathbf{x}_n - \hat{\mathbf{x}}_n)' \quad (12)$$

Remark 1: This set of working equations differs from those of the standard discrete Kalman filter only in the presence of the scalar factor s in Eq. (11). In fact with $s = 1$ the solution reduces to the standard Kalman filter. This simple form makes the method computationally attractive.

Remark 2: Another way to view the concept of Eq. (11) is that, at each step, the a priori data are discounted by multiplying the covariance matrix by s, $s > 1$, hence Eq. (11) takes the following form:

$$P_{n+1}|_n = \mathbf{\Phi}(t_{n+1}, t_n)[s\mathbf{P}_n|_n]\mathbf{\Phi}'(t_{n+1}, t_n) + \mathbf{Q}_n$$

5. Derivation

For the sake of brevity the derivation will be first given for the simplified case

$$\mathbf{x}_{n+1} = \mathbf{\Phi}(t_{n+1}, t_n)\mathbf{x}_n \quad (13)$$

$$\mathbf{y}_n = \mathbf{H}_n\mathbf{x}_n + \mathbf{w}_n \quad (14)$$

The results will be subsequently generalized to include $\mathbf{u}$ as in Eqs. (1) and (2).

When Eq. (13) is substituted into Eq. (14) and it is written out for the first n observation points the resulting set of equations can be condensed into

$$\mathbf{z}_n = \mathbf{A}_n\mathbf{x}_n + \mathbf{v}_n \quad (15)$$

where

$$\mathbf{z}_n = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \cdot \\ \cdot \\ \mathbf{y}_n \end{bmatrix}, \quad \mathbf{v}_n = \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \cdot \\ \cdot \\ \mathbf{w}_n \end{bmatrix}, \quad \mathbf{A}_n = \begin{bmatrix} \mathbf{H}_1\mathbf{\Phi}(t_1, t_n) \\ \mathbf{H}_2\mathbf{\Phi}(t_2, t_n) \\ \cdot \\ \cdot \\ \mathbf{H}_n \end{bmatrix} \quad (16)$$

and clearly by Eq. (5)

$$E\mathbf{v}_n\mathbf{v}_n' = \begin{bmatrix} s^{n-1}\mathbf{R}_1 & & & 0 \\ & s^{n-2}\mathbf{R}_2 & & \\ & & \ddots & \\ 0 & & & \mathbf{R}_n \end{bmatrix} = \mathbf{V}_n \quad (17)$$

So that the weighted mean square estimate, $\hat{\mathbf{x}}_n$, of $\mathbf{x}_n$ is obtained by minimizing with respect to $\mathbf{x}_n$ the following:

$$I_n = (\mathbf{A}_n\mathbf{x}_n - \mathbf{z}_n)'\mathbf{V}_n^{-1}(\mathbf{A}_n\mathbf{x}_n - \mathbf{z}_n) \quad (18)$$

Using calculus of extrema

$$\hat{\mathbf{x}}_n = (\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1}\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{z}_n \quad (19)$$

Let us now write the same results after a new observation y_{n+1} is added

$$\mathbf{z}_{n+1} = \mathbf{A}_{n+1}\mathbf{x}_{n+1} + \mathbf{v}_{n+1} \quad (20)$$

and

$$\hat{\mathbf{x}}_{n+1} = (\mathbf{A}_{n+1}'\mathbf{V}_{n+1}^{-1}\mathbf{A}_{n+1})^{-1}\mathbf{A}_{n+1}'\mathbf{V}_{n+1}^{-1}\mathbf{z}_{n+1} \quad (21)$$

where

$$\mathbf{z}_{n+1} = \begin{bmatrix} \mathbf{z}_n \\ \mathbf{y}_{n+1} \end{bmatrix}, \quad \mathbf{v}_{n+1} = \begin{bmatrix} \mathbf{v}_n \\ \mathbf{w}_{n+1} \end{bmatrix}, \quad \mathbf{A}_{n+1} = \begin{bmatrix} \mathbf{A}_n\mathbf{\Phi}(t_n, t_{n+1}) \\ \mathrm{H}_{n+1} \end{bmatrix} \quad (22)$$

and

$$E\mathbf{v}_{n+1}\mathbf{v}_{n+1}' = \begin{bmatrix} s\mathbf{V}_n & 0 \\ 0 & R_{n+1} \end{bmatrix} = \mathbf{V}_{n+1} \quad (23)$$

Now writing Eq. (21) in the form

$$\mathbf{A}_{n+1}'\mathbf{V}_{n+1}^{-1}\mathbf{A}_{n+1}\hat{\mathbf{x}}_{n+1} = \mathbf{A}_{n+1}'\mathbf{V}_{n+1}^{-1}\mathbf{z}_{n+1} \quad (24)$$

and expanding we get

$$[(1/s)\ \mathbf{\Phi}'(t_n,t_{n+1})\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n\mathbf{\Phi}(t_n,t_{n+1}) + \mathbf{H}_n'\mathbf{R}_{n+1}^{-1}\mathbf{H}_n]\mathbf{x}_{n+1} = 1/s\,\mathbf{\Phi}'(t_n,t_{n+1})\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{z}_n + \mathbf{H}_n'\mathbf{R}_{n+1}^{-1}\mathbf{y}_{n+1} \quad (25)$$

It can be shown (Appendix 1) that

$$\mathbf{P}_n|_n = E\{(\mathbf{x}_n - \hat{\mathbf{x}}_n)(\mathbf{x}_n - \hat{\mathbf{x}}_n)'|\mathbf{y}_k,\ 1 \le k \le n\} = (\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1} \quad (26)$$

$$\mathbf{P}_{n+1}|_n = E\{(\mathbf{x}_{n+1} - \hat{\mathbf{x}}_{n+1})(\mathbf{x}_{n+1} - \hat{\mathbf{x}}_{n+1})'|\mathbf{y}_k,\ 1 \le k \le n\} = s\mathbf{\Phi}(t_{n+1},t_n)\mathbf{P}_n|_n\mathbf{\Phi}'(t_{n+1},t_n) \quad (27)$$

$$\begin{aligned}\mathbf{P}_{n+1}|_{n+1} &= E\{(\mathbf{x}_{n+1} - \hat{\mathbf{x}}_{n+1})(\mathbf{x}_{n+1} - \hat{\mathbf{x}}_{n+1})'|\mathbf{y}_k,\ 1 \le k \le n+1\} \\ &= [s\mathbf{\Phi}'(t_n,t_{n+1})\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n\mathbf{\Phi}(t_n,t_{n+1}) + \mathbf{H}_n'\mathbf{R}_{n+1}^{-1}\mathbf{H}_n]^{-1} \\ &= [\mathbf{P}_{n+1}^{-1}|_n + \mathbf{H}_n'\mathbf{R}_{n+1}^{-1}\mathbf{H}_n]^{-1} \quad (28)\end{aligned}$$

From Eq. (28) also

$$\mathbf{P}_{n+1}|_{n+1} = \mathbf{P}_{n+1}|_n - \mathbf{P}_{n+1}|_n\mathbf{H}_n' \times (\mathbf{R}_{n+1} + \mathbf{H}_n\mathbf{P}_{n+1}|_n\mathbf{H}_n')^{-1}\mathbf{H}_n\mathbf{P}_{n+1}|_n \quad (29)$$

This is easily proven by direct substitution of Eqs. (28) and (29) into

$$\mathbf{P}_{n+1}^{-1}|_{n+1}\mathbf{P}_{n+1}|_{n+1} = \mathbf{I} \quad (30)$$

Utilizing these notations in Eq. (25) we obtain

$$\hat{\mathbf{x}}_{n+1} = \mathbf{\Phi}(t_{n+1},t_n)\hat{\mathbf{x}}_n + \mathbf{P}_{n+1}|_n\mathbf{H}_n'\ (\mathbf{R}_{n+1} + \mathbf{H}_n\mathbf{P}_{n+1}|_n\mathbf{H}_n')^{-1}\ [\mathbf{y}_{n+1} - \mathbf{H}_n\mathbf{\Phi}(t_{n+1},\ t_n)\hat{\mathbf{x}}_n] \quad (31)$$

Remark: The presence of an independent noise term $\mathbf{u}_n$ on the right side of Eq. (13) as in Eq. (1), is easily shown to result simply in the appearance of an additive term $\mathbf{Q}_n$ on the right hand side of Eq (27) and in all $P_{n+1}|_n$ terms in Eq. (29) and Eq. (31) accordingly. This is obvious from the fact that the expectation in Eq. (27) is extrapolated from data available at t_n so the addition of $\mathbf{u}_n$ which is independent of the observations up to t_n will simply increase the covariance of the extrapolated $\mathbf{x}$ by $\mathbf{Q}_n$.

6. Linearized Filtering—Identification

The Kalman filter is widely used as a linearized approximation for plants which are quite nonlinear as well as time varying. In such a situation the coefficients in the signal model equation are purely local derivatives which are usually only vaguely known. This leads to a situation where in the following problem

$$\mathbf{x}_{n+1} = \mathbf{\Phi}_n\mathbf{x}_n + \mathbf{u}_n \quad (32)$$

$$\mathbf{y}_n = \mathbf{H}_n\mathbf{x}_n + \mathbf{w}_n \quad (33)$$

$\mathbf{x}$ and $\mathbf{\Phi}$ are both unknown.

One approach is to use some identification scheme for $\mathbf{\Phi}$. There are vast numbers of proposed techniques for this; most of them require batch processing of both input and output data, of which the input may not be available in the situation considered here.

A filtering approach, using the proposed filter equations and of course only the output is discussed here. Assume that it is known that

$$\frac{||\mathbf{\Phi}(t_{n+1},t_n) - \mathbf{\Phi}(t_{k+1},t_k)||}{||\mathbf{\Phi}(t_{n+1},t_n)||} \ll 1,\ n - k \le m \text{ for some } m \quad (34)$$

Now select a weighting number s such that $(1/s^m) \ll 1$ and augment the state Eq. (32) by

$$\mathbf{\Phi}_{n+1} = \mathbf{\Phi}_n, \quad \mathbf{x}_n \text{ and } \mathbf{\Phi}_n \text{ unknown} \quad (35)$$

Since the selected weighting factor s will effectively suppress information taken outside of the time interval m within which $\mathbf{\Phi}$ indeed is essentially constant, the consistency of the model with reality is preserved.

Now filtering with weighting s can be used on Eqs. (32, 33 and 35), however the state Eqs. (32) and (35) are nonlinear in $\mathbf{x}$ and $\mathbf{\Phi}$. In keeping with the other approximations introduced so far it is logical to linearize Eqs. (32) and (35) around the current estimate $\hat{\mathbf{x}}$ and $\hat{\mathbf{\Phi}}$. This will be illustrated for the second degree case with one unknown parameter, a_n

$$\begin{bmatrix} x_{n+1}^1 \\ x_{n+1}^2 \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} a_n & -0.45 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \\ a_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_n \quad (36)$$

$$y_n = [1 \quad 0 \quad 0] \begin{bmatrix} x_n^1 \\ x_n^2 \\ a_n \end{bmatrix} + w_n \quad (37)$$

$$Eu_n = Ew_n = 0, \quad E\{u_k^2|t = t_n\} = q, \quad E\{w_k^2|t = t_n\} = s^{n-k}r, \quad n \ge k \quad (38)$$

Now linearizing the $a_nx_n^1$ term around the current estimates $\hat{x}_n^1$ and $\hat{a}_n$ we get

$$\mathbf{z}_{n+1} = \mathbf{\Phi}(t_{n+1},\ t_n)\mathbf{z}_n + \mathbf{c}u_n - \mathbf{b}_n \quad (39)$$

$$y_n = \mathbf{h}\mathbf{z}_n + w_n$$

where

$$\mathbf{z}_{n+1} = \begin{bmatrix} x_{n+1}^1 \\ x_{n+1}^2 \\ a_{n+1} \end{bmatrix}, \quad \mathbf{\Phi}(t_{n+1},\ t_n) = \begin{bmatrix} \hat{a}_n & -0.45 & \hat{x}_n^1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (40)$$

$$\mathbf{b}_n = \begin{bmatrix} \hat{a}_n\hat{x}_n^1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{h} = [1 \quad 0 \quad 0] \quad (41)$$

By the direct application of the previous results we have

$$\hat{\mathbf{z}}_{n+1} = \mathbf{\Phi}(t_{n+1},t_n)\hat{\mathbf{z}}_n - \mathbf{b}_n + \mathbf{P}_{n+1}|_n\mathbf{h}'(r + \mathbf{h}\mathbf{P}_{n+1}|_n\mathbf{h}')^{-1} \times \{y_{n+1} - \mathbf{h}[\mathbf{\Phi}(t_{n+1},t_n)\mathbf{z}_n - \mathbf{b}_n]\} \quad (42)$$

$$\mathbf{P}_{n+1}|_{n+1} = \mathbf{P}_{n+1}|_n - \mathbf{P}_{n+1}|_n\mathbf{h}' \times (r + \mathbf{h}\mathbf{P}_{n+1}|_n\mathbf{h}')^{-1}\mathbf{h}\mathbf{P}_{n+1}|_n \quad (43)$$

$$\mathbf{P}_{n+1}|_n = s\mathbf{\Phi}(t_{n+1},\ t_n)\mathbf{P}_n|_n\mathbf{\Phi}'(t_{n+1},\ t_n) + q\mathbf{c}\mathbf{c}' \quad (44)$$

Remark: The effect of the term $\mathbf{b}_n$ in Eq. (39) at each time instant is to insert a constant bias into the value of $\mathbf{z}$ at the next time instant. Thus, if the estimated value of $\mathbf{z}$ at t_n is $\hat{\mathbf{z}}_n$ then the extrapolated estimate of $\mathbf{z}$ at t_{n+1} is simply $\mathbf{\Phi}(t_{n+1},\ t_n)\hat{\mathbf{x}}_n - \mathbf{b}_n$. Consequently the filter equation has the form of Eq (42) instead of Eq. (31).

7. An Abstract Example and Parametric Study

Two such types of experiments are reported here. In the first type (Experiments 1 and 2) the filter uses a model of the correct order but with one unknown parameter. The filter is used to estimate both the state and the unknown parameter. In the second type (Experiment 3) the system is one higher order than its model used in the filter. These two types represent the most common causes of difficulties with Kalman filters.

Type 1. Unknown, Slowly Changing System Parameters

The system described in Eqs. (39–40) can be used in a computer experiment to demonstrate through a parametric

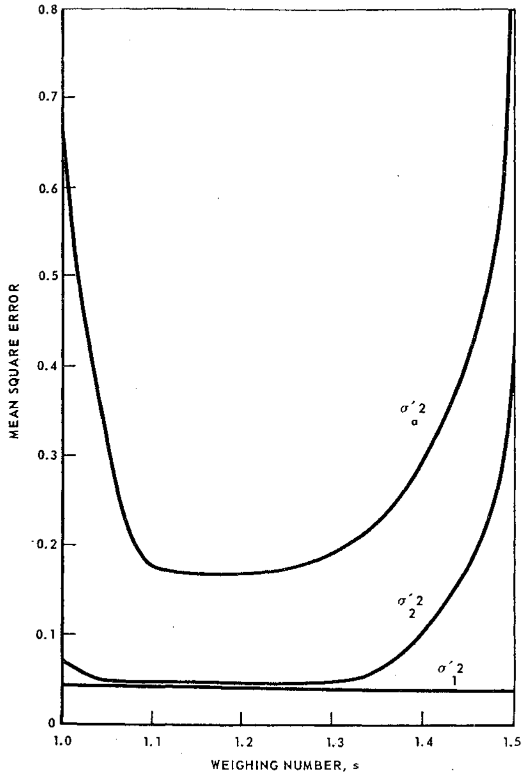

Fig. 1 Effect of the weighing number s on the mean-square estimation error for a system containing one sectionally constant random parameter; $q = 0.1$, $r = 0.05$.

study the effectiveness of the proposed filter. To provide a basis for such an evaluation one has to somehow standardize the actual variation of the unknown parameter a_n.

Experiment 1

The true parameter a is held constant by interval over a sequence of twenty time intervals. The length of the intervals is picked from a Poisson distribution with average length $\lambda = 30$, the amplitude a, within each interval is picked from a normal density $N(0,1)$.

The estimate is improving in time on the average following each change in the value of a so that it should reach its best value at the end of each interval just before switching the value of a. Accordingly, the values used for comparison purposes and plotted in Figs. 1 and 2 are

$$\sigma_a'^2 = \frac{1}{20}\sum_{i=1}^{20}(a_i - \hat{a}_i)^2 \tag{45}$$

$$\sigma_1'^2 = \frac{1}{20}\sum_{i=1}^{20}(x_i^1 - \hat{x}_i^1)^2 \tag{46}$$

$$\sigma_2'^2 = \frac{1}{20}\sum_{i=1}^{20}(x_i^2 - \hat{x}_i^2)^2 \tag{47}$$

where i in the summations refers to the last sample on each interval before the changing of the value of a.

These average error curves in Figs. 1 and 2 show a flat minimum for $s = 1.1 \sim 1.3$ (note $1.1^{-30} = 0.0573$, $1.3^{-30} = 0.00038$). As compared to the Kalman filter at $s = 1$ these minima represent, respectively, a 72% improvement in a, a 49% improvement in x^2, and a 7% improvement in x^1 for plant noise $q = 0.1$ as shown in Fig. 1. Similar results appear for $q = 0.5$ in Fig. 2. The relatively small improvement in x^1 is due to the fact that this state variable is directly measured subject only to the measurement noise. Consequently finding this is a simple smoothing problem which does not really involve the dynamics of the system. On the other hand a and x_2 are not directly measured but derived from measurements of x^1. This effect would become more pronounced for higher dimensional systems. As s increases beyond 1.3 in this example the quality of the filter is again deteriorating since then only a small and decreasing number of the last few samples are actually being used in the estimation. Beyond about $s = 2$ the estimate is really only based on the last sample.

Experiment 2

The true parameter a is changed linearly in time with different rate by interval over a sequence of twenty time intervals. The length of the intervals again is picked from a Poisson distribution with average length $\lambda = 30$, the rate of change of a within each interval is picked from a normal density $N(0, 0.01)$. The corresponding results as in Experiment 1 are plotted in Fig. 3. The conclusions are similar to those in Experiment 1.

Type 2. Lower-Order Filter

In this study the signals were generated over a period of 100 time units by a model of the third order. Specifically

$$\begin{bmatrix} x_{n+1}^1 \\ x_{n+1}^2 \\ x_{n+1}^3 \end{bmatrix} = \begin{bmatrix} 1.96 & -0.9604 & 0.0003 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \\ x_n^3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_n \tag{48}$$

$$y_n = [1 \quad 0 \quad 0] \begin{bmatrix} x_n^1 \\ x_n^2 \\ x_n^3 \end{bmatrix} + w_n$$

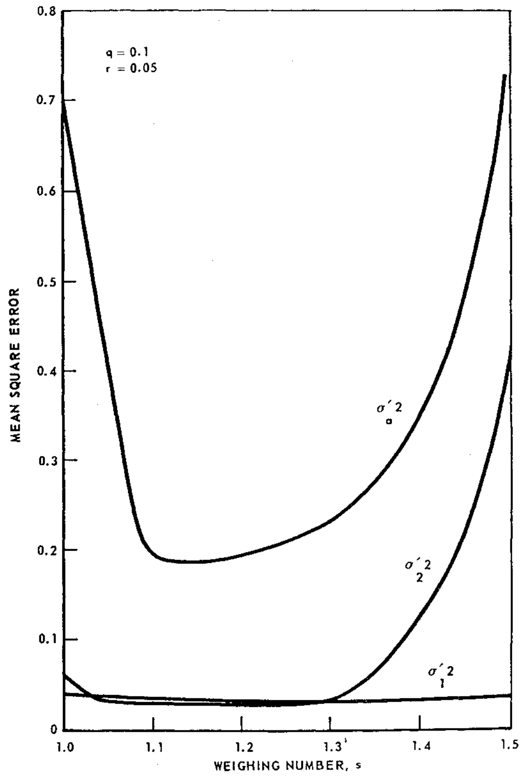

Fig. 2 Effect of the weighing number s on the mean-square estimation error for a system containing one sectionally constant random parameter; $q = 0.5$, $r = 0.05$.

The filter however was constructed for the second degree model:

$$\begin{bmatrix} x_{n+1}^1 \\ x_{n+1}^2 \end{bmatrix} = \begin{bmatrix} 1.96 & -0.9604 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_n$$

$$y_n = [1 \quad 0] \begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} + w_n \qquad (49)$$

As may be observed in Figs. 4 and 5 the Kalman filter ($s = 1$) keeps diverging at $T = 100$ and has reached a very large error value at that point. On the other hand the proposed modified filter at $s = 1.2$ is quite stable and it has a small mean square error value, although it is higher than σ^2, the theoretical variance as used in the Kalman filter with $s = 1$.

8. An Aerospace Example

A gliding vehicle uses an inertial navigation system for its altitude reference which is subject to the usual long range drift of INS systems and thus requires correction from an independent information source. Such a source is altitude computation from dynamic pressure.

If the aerodynamic normal force coefficient, C_N, is known and the INS outputs are resolved to give the normal force F_N then the aerodynamic dynamic pressure is given by

$$q = F_N/AC_N \qquad (50)$$

where A is the reference aero surface area. If the total earth reference velocity of the vehicle, V, is known then the air density is given by

$$\rho = 2q/V^2 \qquad (51)$$

If the density altitude model is known then density altitude is

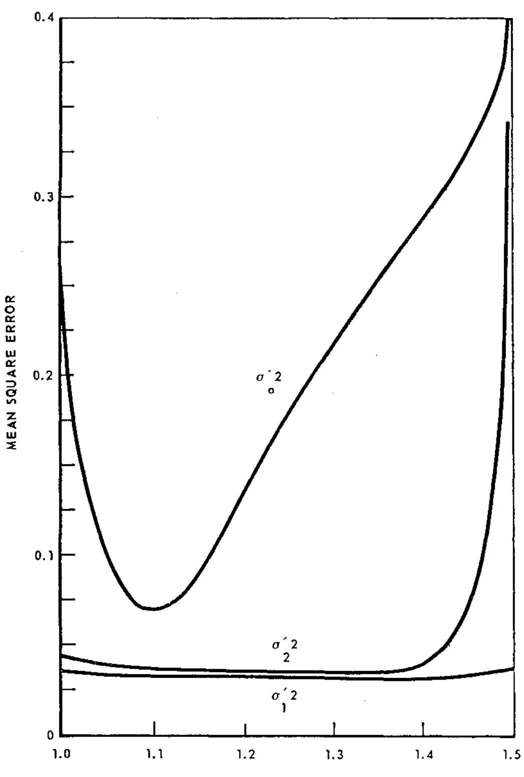

Fig. 3 Effect of the weighing number s on the mean-square estimation error for a system containing one random parameter with sectionally constant time rate; $q = 0.1$, $r = 0.05$.

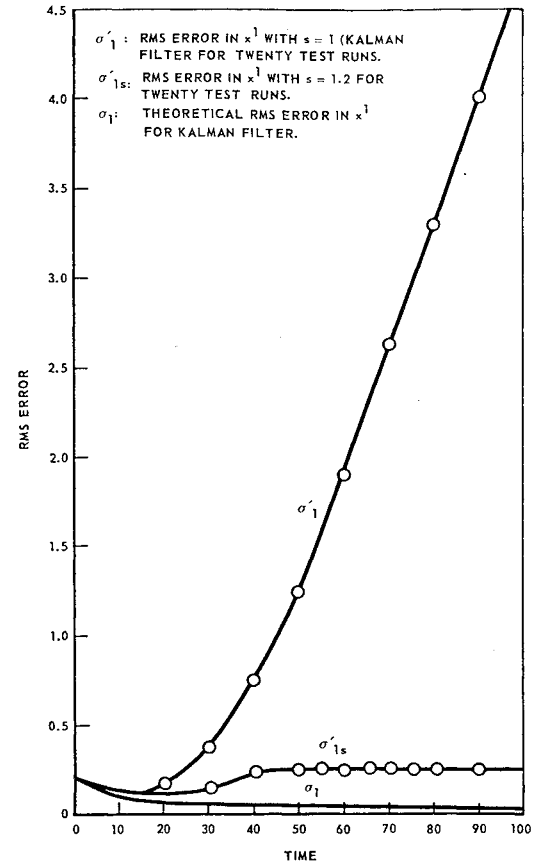

Fig. 4 Effect of weighing $s = 1.2$ on the time response x^1 of second-order filter used for a third-order system; $q = 1.0^{-6}$, $r = 0.05$.

determined from

$$h_\rho = K_1 + K_2 \log\rho + K_3(\log\rho)^2 + K_4(\log\rho)^3 \qquad (52)$$

The error in the INS altitude signal h_{INS} is computed as

$$\delta h = h_\rho - h_{INS}$$

Now h_{INS} is drifting only slowly in level flight when this correction would be carried out and can be considered constant for the period of this determination. On the other hand h_ρ will be quite noisy because of gustiness in the air. In general

$$h_\rho = \hat{h}_\rho + h_{\rho g}(t)$$

where $h_{\rho g}(t)$ is the contribution of the gusts. Note that $h_{\rho g}$ is relatively small so that it will be gaussian in spite of the nonlinearity of Eq. (52) if the gust component in F_N can be considered gaussian. The latter will be assumed to be gaussian with a second degree power spectrum of a cutoff frequency of a. Such a gust component can be represented by a signal model of

$$\dot{h}_\rho = ah_\rho + \alpha v, \quad h_\rho(0) = \hat{h}_\rho(0)$$

where v is white noise and α an amplitude parameter.

Of course both the cutoff frequency a and the noise amplitude α will be varying with the weather cell in which the vehicle is currently flying. If one assumes that the weather has indeed a cellular pattern with something like 50 mile average cell size and that the actual cell size is varying in some random fashion for instance on a Poisson density then one has a situation for the variation of a as studied in the abstract example and the results obtained there are directly applicable provided that digital smoothing is used which might be logical if an onboard computer is available.

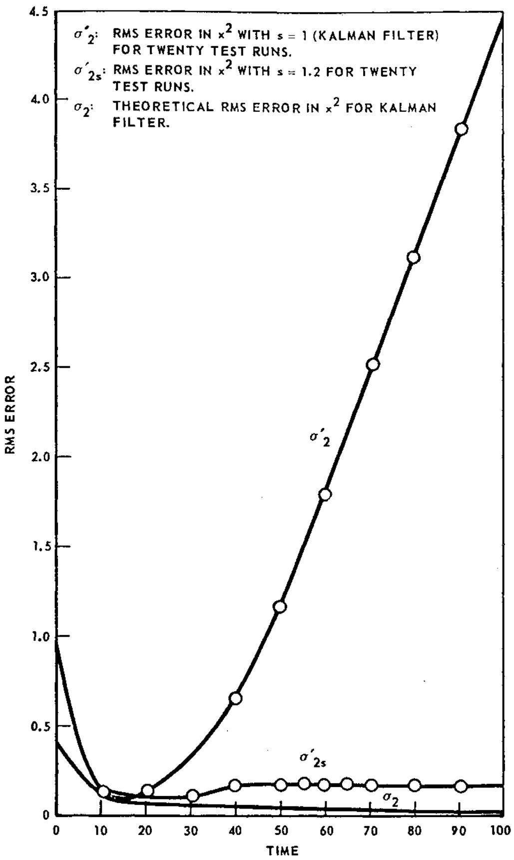

Fig. 5 Effect of weighing $s = 1.2$ on the time response, x^2, of second order filter used for a third order system; $q = 1.0^{-6}$, $r = 0.05$.

A higher degree problem results if it is necessary to consider the dynamics of the accelerometer of the vehicle.

A related higher degree problem is represented by the fairly common practice of correcting INS drift through comparison with some steady state—accurate device like Loran. This problem has typically one to two dozen state variables but otherwise follows the pattern outlined here.

Conclusion

A method is presented to remedy the divergence problem of the regular Kalman filter by weighing the past observation by a progressively small number. The result is a filter which differs from the Kalman filter in one single additional multiplication by a fixed scalar s at each observation time. This shows the computational attractiveness of the proposed algorithm.

Numerical results from computer experiment type parametric studies show very substantial improvements both in filter stability and in the accuracy of the estimates which can be obtained by using a weighing factor $s > 1$ in place of the $s = 1$ value of the regular Kalman filter. The improvement is especially pronounced on such state variables which are not directly measured. The factor s is selected empirically typically from the range $1 \leq s \leq 1.5$.

Appendix

We have

$$\mathbf{x}_{n+1} = \mathbf{\Phi}(t_{n+1},t_n)\mathbf{x}_n \tag{A1}$$

$$\mathbf{y}_n = \mathbf{H}_n\mathbf{x}_n + \mathbf{w}_n \tag{A2}$$

and from Eqs. (15) and (19) we have

$$\mathbf{A}_n\mathbf{x}_n = \mathbf{z}_n - \mathbf{v}_n \tag{A3}$$

$$\hat{\mathbf{x}}_n = (\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1}\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{z}_n \tag{A4}$$

Multiplying (A3) by $(\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1}\mathbf{A}_n'\mathbf{V}_n^{-1}$ and using the fact of (A4) we get

$$\mathbf{x}_n = \hat{\mathbf{x}}_n - (\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1}\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{v}_n \tag{A5}$$

Hence

$$\tilde{\mathbf{x}}_n = \hat{\mathbf{x}}_n - \mathbf{x}_n = (\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1}\mathbf{A}'_n\mathbf{V}_n^{-1}\mathbf{v}_n \tag{A6}$$

Now the covariance matrix of $\mathbf{x}_n$ given all the information up to time t_n will be

$$E\{\tilde{\mathbf{x}}_n\tilde{\mathbf{x}}_n'|\mathbf{y}_k,\ 1 \leq k \leq n\} =$$

$$E\{(\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1}\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{z}_n\mathbf{z}_n'\mathbf{V}_n^{-1\prime}\mathbf{A}_n(\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1\prime}\}$$

$$= (\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1}(\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)'(\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1\prime}$$

$$= (\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n)^{-1} = \mathbf{P}_n|_n \tag{A7}$$

which is Eq. (26).

Similarly we have

$$E\{\tilde{\mathbf{x}}_{n+1}\tilde{\mathbf{x}}_{n+1}'|\mathbf{y}_k,\ 1 \leq k \leq n+1\} =$$

$$(\mathbf{A}_{n+1}'\mathbf{V}_{n+1}^{-1}\mathbf{A}_{n+1})^{-1} = \mathbf{P}_{n+1}|_{n+1}$$

$$= [s\mathbf{\Phi}'(t_n,t_{n+1})\mathbf{A}_n'\mathbf{V}_n^{-1}\mathbf{A}_n\mathbf{\Phi}(t_n,t_{n+1}) + \mathbf{H}_n'\mathbf{R}_{n+1}^{-1}\mathbf{H}_n]^{-1} \tag{A8}$$

which is Eq. (28).

From Eq. (A1) and the argument about the obsolescence of old measurements as explained elsewhere it is clear that

$$E\{\tilde{\mathbf{x}}_{n+1}\tilde{\mathbf{x}}_{n+1}'|\mathbf{y}_k,\ 1 \leq k \leq n\} =$$

$$s\mathbf{\Phi}(t_{n+1},t_n)\mathbf{P}_n|_n\mathbf{\Phi}'(t_{n+1},t_n) \tag{A9}$$

which is Eq. (27).

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